The strength of Nonstandard Analysis Proceedings of the Conference on Nonstandard Mathematics, Aveiro 2004

Imme van den Berg, Vítor Neves, editors

The Date

ABSTRACT Abstracts of the contributions to the Proceedings of the Conference Nonstandard Mathematics 2004, held in Aveiro, Portugal

Contents

Foundations

1 The Strength of Nonstandard Analysis

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Abstract:
A weak theory nonstandard analysis, with types at all finite levels over both the integers and hyperintegers, is developed as a possible framework for reverse mathematics. In this weak theory, we investigate the strength of standard part principles and saturation principles which are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers.

24 p.

2 The Virtue of Simplicity

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Abstract

It is known that IST (internal set theory) is a conservative extension of ZFC (Zermelo-Fraenkel set theory with the axiom of choice); see for example the appendix to [2] for a proof using ultrapowers and ultralimits. But these semantic constructions leave one wondering what actually makes the theory work—what are the inner mechanisms of Abraham Robinson's new logic. Let us examine the question syntactically. Notational conventions: we use x to stand for a variable and other lowercase letters to stand for a sequence of zero or more variables; variables with a prime $'$ range over finite sets; variables with a tilde $\tilde{}$ range over functions. We take as the axioms of IST the axioms of ZFC together with the following, in which A is an internal formula: (T) $\forall^{\mathrm{st}}\mathrm{t}[\forall^{\mathrm{st}}\mathrm{x}\mathrm{A} \to \forall \mathrm{x}\mathrm{A}]$, where A has free variables x and the variables of t, (I) $\forall^{\mathrm{st}}\mathrm{y}'\exists\mathrm{x}\forall\mathrm{y}{\in}\mathrm{y}'\mathrm{A} \leftrightarrow \exists\mathrm{x}\forall^{\mathrm{st}}\mathrm{y}\mathrm{A}$, (S) $\forall^{\mathrm{st}}\mathrm{x}\exists^{\mathrm{st}}\mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y}) \to \exists^{\mathrm{st}}\mathrm{y}\forall^{\mathrm{st}}\mathrm{x}\mathrm{A}\big(\mathrm{x},\tilde{\mathrm{y}}(\mathrm{x})\big)$. We have written the standardization principle (S) in functional form and required A to be internal; we call this the *restricted* standardization principle. It can be shown that the general standardization principle is a consequence. All functions must have a domain. There is a neat way, using the reflection principle of set theory, to ensure that ỹ has a domain, but let me avoid discussion of this point. We do not take the predicate symbol *standard* as basic, but introduce it by

$$x \text{ is standard } \leftrightarrow \exists^{\mathrm{st}} y[y = x].$$

In this way $\forall^{\mathrm{st}}$ and $\exists^{\mathrm{st}}$ are new *logical* symbols and (I), (S), (T) are *logical* axioms of Abraham Robinson's new logic.

3 Analysis of various practices of referring in classical or non standard mathematics

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Abstract.

The thesis underlying my paper is that the various approaches of mathematics, both the conventional or the diverse non standard approaches, pure or applied, are characterized primarily by their mode of referring and in particular by the more or less important use of the *reconstructed* reference, the reference to the sets and collections, that I will distinguish from the *direct* reference, the reference to the world of the facts in a broad sense. The direct reference, in traditional mathematics as well as in non standard mathematics (for the main part) is ritually performed in the classical form of modelling, consisting in confronting the facts to a small paradise (a set) correctly structured. So the discourse on the model acts like a metaphor of the modelized reality.
I will defend another approach, the relative approach, which consists in using the mathematical languages like genuine languages of communication, referring directly to the facts, but accepting the usage of the metaphor of sets (desmasked as such) too strongly culturally established. My matter will be illustrated by the presentation of

1. A mathematical framework for Dirac's calculus .

2. Heaviside calculus with no laplace transform.

The first one starts with a semantical analysis of Dirac's article introducing Delta function. Dirac said : << δ is improper, δ' is more improper>>. If we give it the meaning : <<δ is not completely known, an δ' is less known>>, it works.

So it is necessary to translate it in the language of Relative Mathematics. We know that the non relative attitude consists in the formal affirmation that a model of the delta function is perfectly determined in a paradise, a space of generalized functions. The continuation of the metaphor requires to provide the paradise with a topology. In the relative approach, we prefer to explore more deeply the concepts of a point, infinity, equality with words of relative mathematics.

Finally the oppositionStandard/Nonstandard is s replaced by the antagonism Relative/Non Relative.

12.p

4 Stratified Analysis?

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Abstract:
It is now over forty years since Abraham Robinson realized that "*the concepts and methods of Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers*" (Robinson's Nonstandard Analysis, Introduction, p. 2). The magnitude of Robinson's achievement cannot be overstated. Not only does his framework allow rigorous paraphrases of many arguments of Leibniz, Euler and other mathematicians from the classical period of calculus; it has enabled the development of entirely new, important mathematical techniques and constructs not anticipated by the classics. Researchers working with methods of nonstandard analysis have discovered new significant results in diverse areas of pure and applied mathematics, from number theory to mathematical physics and economics.

It seems fair to say, however, that acceptance of "nonstandard" methods by the larger mathematical community lags far behind their successes. In particular, the oft-expressed hope that infinitesimals would now replace the notorious ϵ-δ method in teaching calculus remains unrealized, in spite of notable efforts by Keisler , Stroyan , Benci and Di Nasso, and others. Sociological reasons—the inherent conservativity of the mathematical community, lack of a concentrated effort at proselytizing—are often mentioned as an explanation. There is also the fact that "nonstandard" methods seem to require heavier reliance on formal logic than is customary in mathematics at large. While acknowledging much truth to all of the above, here I shall concentrate on another contributing difficulty. At the risk of an overstatement, it is this: while it is undoubtedly possible to do calculus by means of infinitesimals in the Robinsonian framework, it does *not* seem possible to do calculus *only* by means of infinitesimals in it.

In Section I examine this shortcoming in detail, review earlier relevant work, and propose a general plan for extending the Robinsonian framework with a goal of remedying this problem and—possibly—diminishing the need for formal logic as well. Section contains a few examples intending to illustrate how mathematical arguments can be conducted in this extended framework. Section presents an axiomatic system in which the techniques of Section can be formalized.

12 p.

5 ERNA at work

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Abstract:

Elementary Recursive Nonstandard Analysis, in short ERNA, is a constructive system of nonstandard analysis proposed around 1995 by Chuaqui, Suppes and Sommer. It has been shown to be consistent and, without standard part function or continuum, it allows major parts of analysis to be developed in an applicable form. We briefly discuss ERNA's foundations and prove some basic results useful for developing calculus.

6 Neutrices in more dimensions

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Abstract:

Neutrices are convex subgroups of the nonstandard real number system, most of them are external sets. They may also be viewed as modules over the external set of al limited numbers, usually non-noetherian.

Because of the convexity and the invariance under some translations and multiplications, the external neutrices are thougth to be appropriate models of orders of magnitudes of numbers.

Using their strong algebraic structure a calculus external numbers has been developped, which includes solving of equations, and even an analysis, for the structure of external numbers has a property of completeness.

This paper contains a further step, towards linear algebra. We show that in $\mathbb{R}^2$ every neutrix is the direct sum of two neutrices of $\mathbb{R}$. The components may be chosen orthogonal, so as to present the length and width of the neutrix. We indicate how to extend the decomposition to spaces of dimension k, with standard k.

22 p.

Number Theory

1 Nonstandard Methods for Additive and Combinatorial Number Theory—A Survey

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From the introduction:

In this article my research on the subject described in the title is summarized. I am not the only person who has worked on this subject. For example, several interesting articles by Steve Leth were published around 1988. I would like to apologize to the reader that no efforts have been made by the author to include other people's research.

My research on nonstandard analysis started when I was a graduate student in the University of Wisconsin. A large part of my thesis was devoted towards solving the problems posed in Keisler and Leth's paper Meager sets on the hyperfinite time line in The Journal of Symbolic Logic. By the time when my thesis was finished, many of the problems had been solved. However, some of them were still open. It took me another three years to find a solution to them. Before this my research on nonstandard analysis was mainly focused on foundational issues concerning the structures of nonstandard universes. After I told Steve about my solutions, he immediately informed me how it could be applied to obtain interesting results in combinatorial number theory. This opened a stargate in front of me and lead me into a new and interesting field.

16. p

2 Nonstandard methods and the Erdös-Turán conjecture

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From the introduction:

A major open question in combinatorial number theory is the Erdős-Turán conjecture which states that if $A =< a_n >$ is a sequence of natural numbers with the property that $\sum_{n=1}^{\infty} 1/a_n$ diverges then A contains arbitrarily long arithmetic progressions. The difficulty of this problem is underscored by the fact that a positive answer would generalize Szemerédi's theorem which says that if a sequence $A \subset \mathbb{N}$ has positive upper Banach Density then A contains arbitrarily long arithmetic progressions. Szemerédi's theorem itself has been the object of intense interest since first conjectured, also by Erdős and Turán, in 1936. First proved by Szemerédi in 1974, the theorem has been re-proved using completely different approaches by Furstenberg in 1977 and Gowers in 1999 , with each proof introducing powerful new methods.

The Erdős-Turán conjecture immediately implies that the primes contain arbitrarily long arithmetic progressions, and it was thought by many that a successful proof for the primes would be the result of either a proof of the conjecture itself or significant progress toward the conjecture. However, very recently Green and Tao were able to solve the question for the primes without generalizing Szemerédi's result in terms of providing weaker density conditions on a sequence guaranteeing that it contain arithmetic progressions.

In this paper we outline some possible ways in which nonstandard methods might be able to provide new approaches to attacking the Erdős-Turán conjecture, or at least other questions about the existence of arithmetic progressions.

Statistics, Probability and Measures

1 Nonstandard likelihood ratio test in exponential families

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Abstract :

Let $(p_\theta)_{\theta\in\Theta}$ be an exponential family in $\mathbb{R}^k$. After establishing nonstandard results about large deviations of the sample mean $\overline{X}$, this paper defines the nonstandard likelihood ratio test of the null hypothesis $H_0 : \theta \in \mathrm{hal}(\widetilde{\Theta}_0)$, where $\widetilde{\Theta}_0$ is a standard subset of Θ and $\mathrm{hal}(\widetilde{\Theta}_0)$ its halo. To get a coherent test the size α must be an infinitesimal number and depending on whether $\frac{\ln \alpha}{n}$ is infinitesimal or not we obtain different rejection criteria. We calculate risks of the first and second kinds (external probabilities) and prove that this test is more powerful than any "regular" nonstandard test based on $\overline{X}$.

Keywords: exponential families, large deviations, asymptotic tests, likelihood ratio test, Bahadur deficiency, nonstandard analysis, external calculus.

2 A Finitary Approach for the Representation of the Infinitesimal Generator of a Markovian Semigroup

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Abstract:

This work is based on Nelson's paper [**?**], where the central question was : under suitable regularity conditions, what is the form of the infinitesimal generator of a Markov semigroup?
In the elementary approach using IST [**?**], the idea is to replace the continuous state space, such as $\mathbb{R}$ with a finite state space X probably containing an unlimited number of points. The topology on X arises naturally from the probability theory. For $x \in X$, let $\mathcal{I}_x$ be the set of all $h \in \mathcal{M}$ vanishing at x where $\mathcal{M}$ is the multiplier algebra of the domain $\mathcal{D}$ of the infinitesimal generator. Define $Ah(x) = \sum_{y \in X \setminus \{x\}} a(x,y)\ h(y)$. To describe the structure of the semigroup generator A, we want to split $Ah(x)$ such that the contribution of the points far from x in X, appears separately. A definition to the quantity $\alpha_{ah}(x) = \sum_{y \in F} a(x,y)\ h(y)$ is given using the least upper bound of the sums on all internal sets W included in the external set F. This leads to give the characterisation of the global part of the infinitesimal generator.

7 p.

3 On two recent applications of nonstandard analysis to the theory of financial markets

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Abstract:

Suitable notions of "unfairness" that measure how far an empirical discounted asset price process is from being a martingale are introduced for complete and incomplete market settings. Several limit processes are involved each time, prompting a nonstandard approach to the analysis of this concept. This leads to an existence result for a "fairest price measure" (rather than a martingale measure) for an asset that is simultaneously traded on several stock exchanges. This approach also proves useful when describing the impact of a currency transaction tax.

15 p.

4 Quantum Bernouilli Experiments and Quantum Stochastic Processes

Manfred Wolff

15 p.

5 A Radon-Nikodým theorem for a vector-valued reference measure

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Abstract:

The conclusion of a Radon-Nikodým theorem is that a measure μ can be represented as an integral with respect to a reference measure such that for all measurable sets A, $\mu(A) = \int_A f_\mu(x)\, d\lambda$ with a (Bochner or Lebesgue) integrable derivative or density f_μ. The measure λ is a usually a countably additive σ-finite measure on the given measure space and the measure μ is absolutely continuous with respect to λ. Different theorems have different range spaces for μ, which could be the real numbers, or Banach spaces with or without the Radon-Nikodým property. In this paper we generalize to derivatives of vector valued measures with respect a vector-valued reference measure. We present a Radon-Nikodým theorem for vector measures of bounded variation that are absolutely continuous with respect to another vector measure of bounded variation. While it is easy in settings such as $\mu << \lambda$, where λ is Lebesgue measure on the interval $[0,1]$ and μ is vector-valued to write down a nonstandard Radon-Nikodým derivative of the form $\varphi : {}^*[0,1] \to \text{fin}({}^*E)$ by $\varphi_\mu(x) = \sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*\lambda(A_i)}\, 1_{A_i}(x)$, a vector valued reference measure does not allow this approach, as the quotient of two vectors in different Banach spaces is undefined. Furthermore, generalizing to a vector valued control measure necessitates the use of a generalization of the Bartle integral, a bilinear vector integral.

11 p.

6 Differentiability of Loeb Measures

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Abstract:

We give a general definition of S-differentiability of an internal measure and compare different particular cases. We show how S-differentiability of an internal measure yields differentiability of the associated Loeb measure and give examples.

We develop a differentiability concept for Loeb measures in a general setting. The theory of differentiable measures was suggested by Fominas an infinite substitute for the Sobolev-Schwartz theory of distributions and has extended rapidly to a strong field of research. In particular during the last ten years it has been the foundation for many applications in different fields such as quantum field theory or stochastic analysis

10.p

Differential systems and equations

1 The power of Gâteaux differentiability

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Abstract:
The search for useful non standard minimization conditions on C^1 functionals defined on Banach spaces lead us to a very simple argument which shows that if a C^1 function $f : E \longrightarrow F$ beween Banach spaces is actually Gâteaux differentiable on finite points along finite vectors, then it is uniformly continuous on bounded sets if and only if it is lipschitzian on bounded sets. The following is a development of these ideas starting from locally convex spaces.

11p.

2 Nonstandard Palais-Smale conditions

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Abstract:

We present nonstandard versions of the Palais-Smale condition some of them generalizations, but still sufficient to prove Mountain Pass Theorems, which are quite important in Critical Point Theory.

16 p

3 Averaging for Ordinary Differential Equations and Functional Differential Equations

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Abstract:

A nonstandard approach to averaging theory for ordinary differential equations and functional differential equations is developed. We define a notion of perturbation and we obtain averaging results under weaker conditions than the results in the literature. The classical averaging theorems approximate the solutions of the system by the solutions of the averaged system, for Lipschitz continuous vector fields, and when the solutions exist on the same interval than the solutions of the averaged system. We extend these results to all perturbations of a vector field which is continuous in the spatial variable uniformly in time and without any restriction on the interval of existence of the solution.

18p.

4 Path-space measure for stochastic differential equation with a coefficient of polynomial growth

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Abstract:

A σ-additive measure over a space of paths is constructed to give the solution to the Fokker-Planck equation associated with a stochastic differential equation with coefficient function of polynomial growth by making use of nonstandard analysis.

10 p.

5 Optimal control for Navier-Stokes equations

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From the introduction:

In this paper we give a brief survey of recent results concerning the existence of optimal controls for the stochastic Navier-Stokes equations (NSE) in a bounded domain D in 2 and 3 space dimensions; that is, $D \subset \mathbb{R}^d$ with $d = 2$ or 3. The controlled equations in their most general form are as follows:

$$u(t) = u_0 + \int_0^t \{-\nu Au(s) - B(u(s)) + f(s, u(s), \theta(s, u))\} \, ds + \int_0^t g(s, u(s)) \, dw(s)$$

Here the evolving velocity field $u = u(t, \omega)$ is a stochastic process with values in the Hilbert space $\mathbf{H} \subseteq \mathbf{L}^d(D)$ of divergence free functions with domain D; this gives the (random) velocity $u(t, x, \omega) \in \mathbb{R}^d$ of the fluid at any time t and point $x \in D$. The most general kind of control θ that we consider acts through the external forcing term f, and takes the form $\theta : [0, T] \times \mathcal{H} \to M$ where $\mathcal{H}$ is the space of paths in $\mathbf{H}$ and the control space M is a compact metric space. In certain settings however it is necessary to restrict to controls of the form $\theta : [0, T] \to M$ that involve no feedback, or those where the feedback only takes account of the instantaneous velocity $u(t)$.

The terms νA, B in the equations are the classical terms representing the effect of viscosity and the interaction of the particles of fluid respectively; the term B is quadratic in u and is the cause of the difficulties associated with solving the Navier–Stokes equations (even in the deterministic case $g = 0$). The final term in the equation represents noisy external forces, with w denoting an infinite dimensional Wiener process.

The methods involve the Loeb space techniques that were employed by Capiński and Cutland to solve the stochastic Navier–Stokes equations with general force and multiplicative noise (that is, with noise $g(s, u(s))$ involving feedback of the solution u), combined with the nonstandard ideas used earlier in the study of optimal control of finite dimensional equations. For the 3-d case it is necessary to utilize the idea of approximate solutions developed by Cutland and Keisler for the study of attractors

23 p.

6 Local-in-time existence of strong solutions of the n-dimensional Burgers equation via discretizations

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Abstract:

Consider the equation:

$$u_t = \nu\Delta u - (u \cdot \nabla)u + f \quad \text{for} \quad x \in [0,1]^n \;\text{ and }\; t \in (0,\infty),$$

together with periodic boundary conditions and initial condition $u(t,0) = g(x)$. This corresponds a Navier-Stokes problem where the incompressibility condition has been dropped. The only major difficulty in existence proofs for this simplified problem is the unbounded advection term.

We present a proof of local-in-time existence of a smooth solution based on a discretization by a suitable Euler scheme. It will be shown that this solution exists in an interval $[0,T)$, where $T \leq \frac{1}{C}$, with C depending only on n and the values of the Lipshitz constants of f and u at time 0. The argument given is quite direct, and makes it clear that these type of results about T are obtainable by local estimates of the solutions of the discretized problem.

13 p.

Infinitesimals and education

1 Calculus with Infinitesimals

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From the introduction:

Abraham Robinson discovered a rigorous approach to calculus with infinitesimals in 1960 and published it in [?]. This solved a 300 year old problem dating to Leibniz and Newton. Extending the ordered field of (Dedekind) "real" numbers to include infinitesimals is not difficult algebraically, but calculus depends on approximations with transcendental functions. Robinson used mathematical logic to show how to extend all real functions in a way that preserves their properties in a precise sense. These properties can be used to develop calculus with infinitesimals. Infinitesimal numbers have always fit basic intuitive approximation when certain quantities are "small enough," but Leibniz, Euler, and many others could not make the approach free of contradiction. Section 1 of this article uses some intuitive approximations to derive a few fundamental results of analysis. We use approximate equality, $x \approx y$, only in an intuitive sense that "x is sufficiently close to y".

H. Jerome Keisler developed simpler approaches to Robinson's logic and began using infinitesimals in beginning U.S. calculus courses in 1969. The experimental and first edition of his book were used widely in the 1970's. Section 2 of this article completes the intuitive proofs of Section 1 using Keisler's approach to infinitesimals.

20p.

2 Pre-University Analysis

Richard O'Donovan

From the introduction:

Using infinitesimals in an introductory course of analysis at pre-university level requires a simpler background than the one usually found in nonstandard analysis publications. Transfer and star-map are concepts far too complicated to introduce at this level. Conceptual difficulties arise in pedagogical approaches which try to avoid these concepts. How is $f(x + dx)$ defined for transcendental functions? How are the derivative, integral and infinite series defined for nonstandard values when transfer is not available? If too many properties need to be introduced axiomatically, the meaning and power of proof is diminished. Is this an acceptable price?

6p.